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Technical Report No. 92

SACLANT ASW
RESEARCH CENTRE

THE INFLUENCE OF QUANTIZING AND SAMPLING ON THE SIGNAL-TO-NOISE RATIO OF A COMPLEX CORRELATOR

by

T. KOOIJ and R. LAVAL

1 SEPTEMBER 1967

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APPROVED FOR DISTRIBUTION

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THE INFLUENCE OF QUANTIZING AND SAMPLING ON THE SIGNAL-TO-NOISE RATIO OF A COMPLEX CORRELATOR

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ABSTRACT

The combined effects of sampling and phase-quantizing of phase-modulated signals in Gaussian noise are studied and expressed in terms of the output signal-to-noise ratio of a matched filter correlator. The obtained results are the basis for finding the sampling and quantizing losses in other types of correlators or filters. Since only the signal bandwidth, and not the carrier frequency, is considered to be of importance, complex envelopes are used to describe the signals, leading to complex correlator types. The results are summarized in three-dimensional graphs.

LIST OF SYMBOLS

a	attenuation
$A(t)$	cosine component of expected signal
$B(t)$	sine component of expected signal
$c(\tau)$	correlation function of cosine components of received and expected signals
E	expectation, statistical average
F	central frequency of narrow-band signal before demodulating
$h(t)$	weighting function of complex matched filter
$I\left(\frac{aS}{N(t)}\right)$	average length of normalized noisy input vector for given aS and $N(t)$.
$I\left(\frac{a\lvert A(t)\rvert}{\sigma_n}\right)$	probability that clipped component of input signal has the same sign as the same component of the expected signal

$$k = \frac{1}{m} \sum_{-\infty}^{\infty} \frac{\sin \frac{p}{m}\pi}{\frac{p}{m}\pi} \cdot \frac{2}{\pi} \cdot \arcsin \left(\frac{\sin \frac{p}{m}\pi}{\frac{p}{m}\pi}\right) = \text{noise factor}$$

m	sampling rate multiplier ($m = 1$ for sampling with frequency ΔF), or phase quantizing factor ($m = 2$ for $\frac{\pi}{2}$ - phase quantizing)

$n(t)$	input noise signal per component
N	$T \cdot \Delta F$ compression factor; gain factor
$N(t)$	modulus of complex input noise signal
$\overline{N}$	average noise modulus
$p\left[\varphi_1, \varphi_2\right]$	joint probability density function of $\varphi_n(t_1)$ and $\varphi_n(t_2)$
$R_S(\tau)$	time-autocorrelation function of one component of the expected signal
S	constant modulus of expected signal
SNR	signal-to-noise ratio
T	time duration of expected signal
$\alpha(t)$	cosine component of received signal
$\beta(t)$	sine component of received signal
$\gamma(\tau)$	complex correlation function
$\Gamma(\omega)$	Fourier spectrum of complex correlator output
$\delta(t)$	Dirac function
ΔF	signal and noise bandwidths
Δt	sampling interval $= \dfrac{1}{m \cdot \Delta F}$

$\eta(t)$	expected signal $= A(t) + j\,B(t) = S\,e^{j\varphi_s(t)}$
$H(\omega)$	Fourier transform of expected signal
$\nu(t)$	complex input noise signal $= N(t)\,.\,e^{j\varphi_n(t)}$
$\xi(t)$	received signal $= \alpha(t)+j\beta(t) = aS.e^{j\left[\varphi_s(t)+\varphi_o\right]} + \nu(t)$
$\Xi(\omega)$	Fourier transform of input noise
$\frac{\pi}{m}$	phase-quantizing step ($m \geq 2$)
$\rho_n(\tau)$	autocorrelation function of the input noise
σ_s^2	power per expected signal component $= \frac{1}{2}S^2$
σ_n^2	power per input noise component
τ	correlation delay parameter
φ_o	phase difference between received and expected signal
$\varphi_s(t)$	phase angle of expected signal
$\varphi_{n}(t)$	phase angle of complex-input noise

INTRODUCTION

Correlators belong to the class of devices that average the product of two functions. Although for a correlator these functions are signals in time, they have the operations of product-averaging in common with other "signal processors" that mathematically perform the same averaging over a different variable, for example over azimuth in the case of a bearing estimator. All systems deriving decisions from second-order statistics perform basically the same operations as a correlator.

The name "correlator" thus defined varies with the properties of the functions over which the product is averaged. If both functions are completely deterministic, the device is only a calculator, calculating the mean product. For non-triviality we shall require therefore that at least one of the two signals is disturbed by some non-deterministic corruption. If one signal is corrupted and the other not, the latter is called the "filter function" or "system function", or in special circumstances the "weighting function" or "impulse response function". The properties of this second signal define the name of the system, i.e. "matched filter" if the second function resembles the non-corrupted input signal, simply "filter" in the case of an unspecified function, or "frequency analyser" if the second function is a pure sine wave.

Only if both of the averaged signals are disturbed by a random background does the averaging device properly deserve the name "correlator". The table below may help to distinguish between the possible correlator systems.

Type of Correlator	Signal x(t)		Signal y(t)		Averaged of shifted time product $\int x(t)\, y(t-\tau)\, dt$
	Known	Random	Known	Random	
1	x		x		calculator
2		x	x		Filter
3		x	known part of x(t)		matched filter
4		x	$e^{j2\pi ft}$		frequency analyzer
5		x		x	proper correlator

All correlators have in common the need for a memory for both signals involved. In an analogue filter one of the signals is stored in the configuration of the filter components, the other signal (the input) has its running past stored as partial contributions to the electromagnetic fields in the components.

Modern signal-processing has for the last ten years made extensive use of binary digit memories (cores, delay lines, flip-flops). The cost of these memories is generally presented per bit. Reduction of the necessary number of bits therefore is and will be

of increasing economic importance. Digitizing means sampling in time and quantizing in amplitude, and this paper tries to present a systematic approach to describe the effects of economizing on both of these operations when applied to phase-modulated signals corrupted by Gaussian noise, thereby permitting the cost per bit to be weighed against the loss of performance.

Rather than studying the effects of bit-reduction on all systems given in the table above, it has been decided to study the effects on only one signal at a time. This is done by comparing or correlating the quantized signal with its uncorrupted replica. The results of this study are therefore only directly valid for a matched filter, and only valid by implication for the other correlator-types.

Another restriction in this study is the requirement of pure phase-modulation of the signals. This makes the results less general in theory, but in practice many signals applied in systems for active detection and acquisition (e.g. tracking) are of this constant-amplitude, phase-modulated type.

The expression "complex" in the title indicates that each signal has been considered to be demodulated by its central frequency, leaving a "complex envelope" (in practice a pair of in-phase and quadrature components). The correlator for such signals naturally bears the name "complex correlator". If the original phase-modulated signal is demodulated with a frequency different from the carrier, the complex envelope will contain a residual modulation

equal to the difference frequency. Because of the uniform phase distribution of the modulating inter-frequency, the signal-to-noise ratios derived in this paper for the purely complex correlator apply equally well to a correlation system that for technical reasons uses a residual carrier (ac correlator).

A widely used method of amplitude-quantizing is infinite clipping, which destroys all amplitude information and retains only the real axis crossings of the phase. By linear interpolation between the crossings, a phase curve may be reconstructed that will resemble the phase of the original, non-clipped signal. Although there are different approaches of clipped signal reconstruction (e.g. using Fourier transform methods), in this paper the theoretical reconstruction method is to interpolate the phase between the real axis crossings, as indicated above, and to consider this function to be the phase of a signal with a constant modulus, so that the reconstructed signal is purely phase-modulated.

It will be clear that this method is much more unfavourable to amplitude-modulated signals (which are completely stripped of their side-bands by the clipping) than to phase-modulated signals. Gaussian noise has its information more or less equally divided between pure phase and amplitude, so that phase-modulated signals plus noise are in the favoured half.

If the clipping and interpolation process is thought to have happened at a sufficiently narrow relative bandwidth, the reconstructed wave will have a phase curve that closely follows the original phase

law, so that the same will be true for the phase of the complex envelope after demodulation with the carrier. The total effect will be that the complex low frequency signal has been trimmed or "shaved" to a constant, normalized modulus.

This is the only method of modulus quantizing that is considered in this report. It is clear that such a system is not able to provide information about the absolute energy of the input signal, but only about energy ratios with respect to other inputs, such as noise.

The other type of quantizing considered is phase-quantizing of the complex envelope. Although phase-quantizing is equivalent to amplitude-quantizing of the in-phase and quadrature components, these quantizations are not independent, but related to each other as cosine and sine.

It is known that a proper Nyquist sampling frequency (twice the signal bandwidth) is sufficient to describe a signal without loss of information. However, if this same sampling frequency is tentatively used on the same signal after clipping, the losses — due to mismatched sampling frequency as a result of the widened signal bandwidth — will generally be even greater than they already were due to modulus clipping alone.

After an explicit summary of constraints and assumptions, the effects on correlator performance of combined modulus normalizing (clipping the high frequency input), phase-quantizing, and sampling, will be followed step by step.

A summary of the results is given in graphical form in the last chapter.

1. ASSUMPTIONS AND DEFINITIONS

The complex low-frequency signals considered in this report will be assumed to have the following properties:

a. The time duration of the signal is T (seconds).

b. The signal has a flat spectrum with bandwidth ΔF(Hz), centred at zero frequency.

c. The signal has a uniform phase distribution.

d. The expected signal has a constant modulus $S = \sigma_s \sqrt{2}$.

e. The received signal is an attenuated copy of the expected signal, with a constant modulus aS and with a constant phase shift φ_0 with respect to the expected signal.

f. The expected signal is not quantized.

g. The quadrature components of the noise have a flat spectrum (bandwidth ΔF, centred at zero frequency) and a Gaussian amplitude distribution (variance σ_n^2).

h. The input SNR will be defined to be the average signal power divided by the average input noise power.

i. The output SNR will be defined to be the square of the average magnitude of the correlation peak divided by the average output noise power.

2. QUANTIZING AND SAMPLING

2.1 Quantizing

The different types of quantizing of the received signal considered in this report are:

a. Normalizing the modulus to a constant length with conservation of phase information.

b. Phase-quantizing combined with normalizing the modulus. This operation converts the signal to a vector in the complex plane with a fixed magnitude and a finite number of possible phase angles. These angles will usually be chosen regularly, so that a regular vector star may be used to describe this kind of quantizing.

The simplest form of phase quantizing is a vector star consisting of four vectors (Fig. 1).

If the star is orientated at 45° to the horizontal, this form is equivalent to clipping the quadrature components, as may be done in a technical system (Ref. 1). The limiting case of infinite phase-quantizing makes type (b) euqal to type (a): complete phase description and normalized modulus.

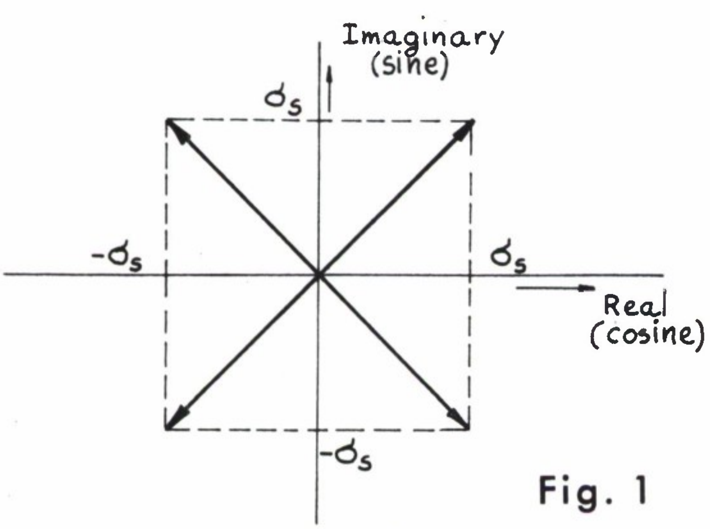

Fig. 1

2.2 Sampling

The sampling theorem states (Ref. 2) that a high-frequency signal with bandwidth ΔF is described completely by double-samples taken at time intervals of $1/\Delta F$ seconds. Each sample determines the amplitude of the signal and its phase with respect to the carrier, and could therefore be represented as a complex sample.

The sampling frequency ΔF is also found by considering that demodulating the high-frequency signal translates the spectrum to zero frequency (Fig. 2). The two, real, low-frequency signals describing the envelope and the phase of the modulating function will have a physical bandwidth of $\frac{1}{2}\Delta F$, thus requiring a sampling frequency of ΔF Hz, as before.

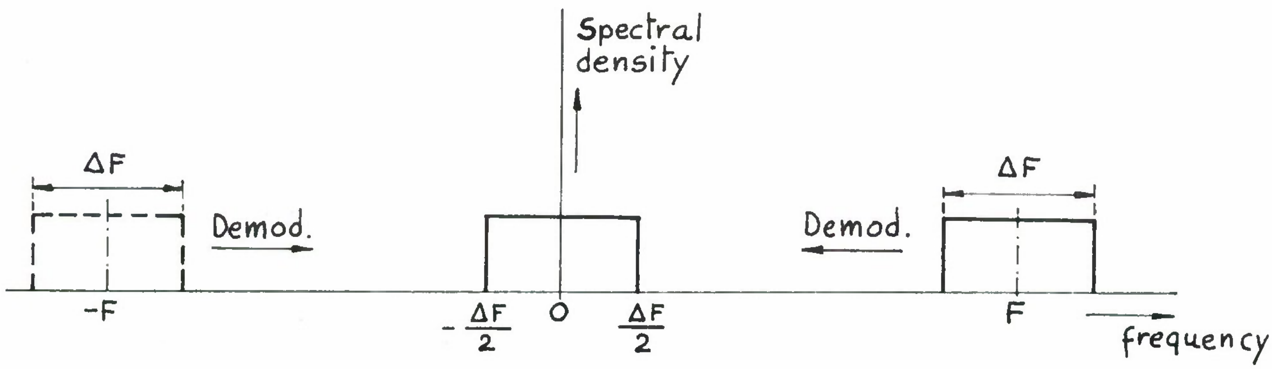

Fig. 2

Quantizing operations — such as clipping the low-frequency quadrature components, which in itself causes loss of information and deterioration of the SNR in detection — tend to broaden the bandwidth of the original signals. The sampling frequency ΔF therefore will no longer be adequate to describe the quantized signals completely, resulting in even more losses in the SNR of a detector.

It is to be expected that the losses will diminish if the sampling frequency is increased. This is seen clearly in the case of the clipping of the quadrature components, which are then, after clipping, described completely by their zero-crossings. These zero-crossings can be determined with a higher accuracy when the sampling frequency is increased. The losses due to the clipping alone are found by increasing the sampling rate to continuous description in time ('infinite sampling').

3. THE REFERENCE SYSTEM

The behaviour of a complex correlator without any quantizing or sampling will be taken as a standard of performance.

With the assumptionsof Chapter 1, and using Greek symbols for complex functions, we consider the complex correlation

$$\gamma(\tau) = 1/T \int \xi(t)\ \eta^{*}(t - \tau)\ dt \qquad \text{(Eq. 1)}$$

where

$$\xi(t) = aS\ e^{i\left[\varphi_s(t) + \varphi_o\right]} + \nu(t) = \text{received signal}$$

$$\eta(t) = S\ e^{i\varphi_s(t)} = \text{expected signal}$$

$$\nu(t) = N(t)\ e^{i\varphi_n(t)} = \text{noise}$$

3.1 Signal Correlation Peak

This correlation function yields for $\tau = 0$ (coincidence):

$$\gamma(0) = \gamma_s(0) + \gamma_n(0) \qquad \text{(Eq. 2)}$$

$$= (aS^2/T)\ e^{i\varphi_o} \int e^{i\left[\varphi_s(t)-\varphi_s(t)\right]}dt + S/T \int e^{-i\varphi_s(t)} N(t) e^{i\varphi_n(t)}\ dt$$

The actual value of $\gamma(0)$ depends on the contribution $\gamma_n(0)$ due to the noise. The mean value of $\gamma(0)$ is found by taking the statistical average

$$E\left[\gamma(0)\right] = \gamma_s(0) + E\left[\gamma_n(0)\right], \qquad \text{(Eq. 3)}$$

where $\gamma_n(0)$ is the only random variable, and

$$E\left[\gamma_n(0)\right] = S/T \int E\left(N(t)\, e^{\left[i\varphi_n(t) - \varphi_s(t)\right]}\right) dt \qquad \text{(Eq. 4)}$$

The integrand is zero, being the average of a vector with uniform phase distribution. Therefore

$$E\left[\gamma_n(0)\right] = 0$$

and

$$E\left[\gamma(0)\right] = \gamma_s(0) = (aS^2T/T)e^{i\varphi_o} = \boxed{2a\sigma_s^{2}\, e^{i\varphi_o}} \qquad \text{(Eq. 5)}$$

It is seen that the initial phase difference φ_o between received and expected signal is repeated in the phase of the correlation peak.

3.2 Noise

The average power of the correlator output if the input is noise only, is found by considering the complex correlator as a matched filter i.e. a filter having a weighting function $h(t)$ equal to the complex

conjugate of the expected signal, running backwards in time:

$$h(t) = (1/T)\ \eta^{*}(-t) \qquad \text{(Eq. 6)}$$

The factor (1/T) is due to the normalizing factor in the correlation (Eq. 1).

The filter frequency response is equal to the complex conjugate of the spectrum of the expected signal. Written as a Fourier pair

$$(1/T)\ \eta^{*}(-t) \rightleftarrows (1/T)\ H^{*}(\omega) \qquad \text{(Eq. 7)}$$

if $\eta(t) \rightleftarrows H(\omega)$ (Ref. 2 , rules 2 & 3)

It is further known (Ref. 3) that the spectral density of the output of a linear filter is equal to the input spectral density multiplied by the squared modulus of the filter response:

$$|\Gamma_n(\omega)|^2 = |\Xi_n(\omega)|^2 (1/T^2)\ |H^{*}(\omega)|^2 \qquad \text{(Eq. 8)}$$

where

$|\Gamma_n(\omega)|^2$ = output noise spectral density

$|\Xi_n(\omega)|^2$ = input noise spectral density

$\frac{1}{T} H^{*}(\omega)$ = matched filter response

The total input power is equal to $2\sigma_n^2$ (two orthogonal noise signals of power σ_n^2, causing the modulus to have a Rayleigh distribution with variance $2\sigma_n^2$ (Ref. 3). This input power is uniformly spread out over ΔF, so that the input density spectrum has a height equal to $2\sigma_n^2/\Delta F$ (Fig. 3).

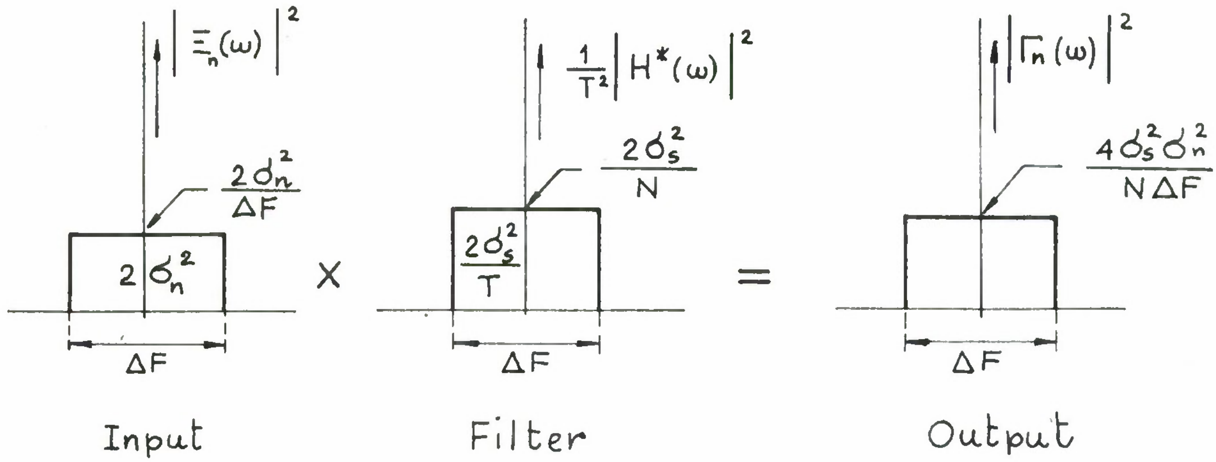

Fig. 3

The filter factor $(1/T^2)\,|H^*(\omega)|^2$ in Eq. 8 is proportional to the energy density spectrum $|H(\omega)|^2$ of the expected signal, having a total energy equal to $S^2T = 2\sigma_s^2\,T$, also uniformly spread out over the bandwidth ΔF. The height of the squared filter response is thus (see Fig. 3):

$$(1/\Delta F)(1/T^2)\,|H^*(\omega)|^2 = (1/T^2)(2\sigma_s^2\,T)/\Delta F = 2\sigma_s^2/T\Delta F = 2\sigma_s^2/N$$

(Eq. 9)

$N = T\ \Delta F$ = compression or gain factor.

The total output power is found by integrating the output density spectrum over ΔF, yielding (see Fig. 3):

$$\sigma^2_{n(out)} = |\Gamma_n(\omega)|^2 \Delta F = (2\sigma_n^2/\Delta F)(2\sigma_s^2/N)\Delta F = \boxed{4\sigma_s^2\sigma_n^2/N} \qquad \text{(Eq. 10)}$$

3.3 Signal-to-Noise Ratio

The output SNR of the correlator, being the ratio between the squared average of the correlation peak (Eq. 5) and the mean output noise power (Eq. 10), becomes

$$(SNR)_{ref} = (4a^2\sigma_s^4)/(4\sigma_s^2\sigma_n^2/N) = \boxed{N\left[(a^2\sigma_s^2)/(\sigma_n^2)\right]} = N(SNR)_{in} \qquad \text{(Eq. 11)}$$

It is seen that the gain of the system is equal to the dispersion or compression factor $N = T.\Delta F$.

If we write the result in Eq. 11 slightly differently, we obtain the well-known relation

$$(SNR)_{ref} = (2a^2\sigma_s^2 T)/(2\sigma_n^2/\Delta F) = \frac{\text{total received signal energy}}{\text{noise power per Hz}}$$

(Eq. 12)

4. CALCULATION OF THE COMPLEX SNR BY REAL AND IMAGINARY PARTS

If the complex functions are described by their real and imaginary parts instead of by the vector notation of the foregoing chapter, the complex correlation of Eq. 1 becomes:

$$\gamma(t)=1/T \int \xi(t)\eta^*(t-\tau)dt=1/T\int \left[\alpha(t)+j\beta(t)\right]\left[A(t-\tau)-jB(t-\tau)\right] dt =$$

$$= 1/T \int (\alpha A + \beta B)\ dt + j\ 1/T \int (\beta A - \alpha B)\ dt \qquad \text{(Eq. 13)}$$

where

$$\left.\begin{aligned} \xi(t) &= \alpha(t) + j\ \beta(t) && = \text{received signal} \\ \eta(t) &= A(t) + j\ B(t) && = \text{expected signal} \end{aligned}\right\} \qquad \text{(Eq. 14)}$$

If φ_o, (the phase difference between received and expected signal), equals zero, α and β will be attenuated copies of their expected counterparts A and B, immersed in Gaussian noise. Only the real part of the complex correlation contributes to the signal correlation, while the imaginary part in Eq. 13 has a mean value of zero because of the orthogonality of β,A and α,B if $\varphi_o = 0$.

Assuming that $1/T \int \alpha A\ dt$ (which, for $\varphi_o = 0$, is the correlation of a real function with its replica) has a SNR equal to $S/\overline{N}$

then the SNR for $1/T \int (\alpha A + \beta B)\, dt$ will be $2S/\overline{N}\sqrt{2}$ because the signal strength is doubled, while the noise power increases only by 3 dB, since the quadrature components, α_{noise} and β_{noise}, are uncorrelated. The same increase (3 dB) in noise power results from the vertical or imaginary contribution of the complete correlation (Eq. 13), while the signal part remains unaltered, if $\varphi_o = 0$. The total SNR thus becomes $2S/\overline{N}\sqrt{2}\sqrt{2} = S/N$, which is equal to the SNR that was assumed for $1/T \int \alpha A\, dt$.

If $\varphi_o \neq 0$, α and β are no longer replicas of A and B. However, the whole system is rotationally invariant in amplitude, due to the assumed uniform phase distributions and the complete phase description of the expected signal. The results found for $\varphi_o = 0$ are therefore generally valid as far as amplitudes are concerned.

EXAMPLE:

The output SNR of the reference system will be determined by the method described above.

a. Signal Correlation Peak

If $\varphi_o = 0$, the cosine component of the input is

$$\alpha(t) = a\, A(t) + n(t) \qquad \text{(Eq. 15)}$$

a = attenuation

A(t) = cosine component of expected signal

n(t) = cosine component of input noise

Correlation with A(t) gives

$$C(\tau) = 1/T \int \left[a\ A(t) + n(t) \right] A(t - \tau)\ dt \qquad \text{(Eq. 16)}$$

In coincidence ($\tau = 0$) the correlation peak is obtained

$$C(0) = C_s(0) + C_n(0) = a/T \int A^2(t)\ dt + 1/T \int n(t)A(t)\ dt \qquad \text{(Eq. 17)}$$

The mean value of the peak is

$$E\left[C(0)\right] = C_s(0) + E\left[C_n(0)\right] \qquad \text{(Eq. 18)}$$

$C_n(0)$ being the only random variable.

$$E\left[C_n(0)\right] = 1/T \int E\left[n(t)\right] A(t)\ dt = 0 \qquad \text{(Eq. 19)}$$

The integrand in Eq. 19 equals zero, because the mean value of the noise is zero.

Thus , using Eq. 17,

$$E\left[C(0)\right] = C_s(0) = a/T \int A^2(t)dt = \frac{a\sigma_s^2\ T}{T} = \boxed{a\sigma_s^2} \qquad \text{(Eq. 20)}$$

where

$\sigma_s^2 T$ = signal energy per component

b. <u>Noise</u>

In the absence of signal, the correlator output at a given instant t_1 equals

$$C_n(t_1) = 1/T \int n(t)\ A(t)\ dt \qquad \text{(Eq. 21)}$$

This integral is a stochastic variable with zero mean (Eq. 19). Its variance is the statistical average of its square:

$$E\left[C_n^2\right] = 1/T^2 \int_{t_1} \int_{\theta} E\left[n(t)\ A(t)\ n(\theta)\ A(\theta)\right] dt\ d\theta =$$

$$= 1/T^2 \int \int A(t)\ A(\theta)\ E\left[n(t)\ n(\theta)\right] dt\ d\theta \qquad \text{(Eq. 22)}$$

where

$$E\left[n(t)\ n(\theta)\right] = \int \int n(t)\, n(\theta)\ \mathrm{prob}\left[n_t, n_\theta\right] dt\, d\theta = \rho_n(t-\theta) = \rho_n(\tau) \qquad \text{(Eq. 23)}$$

in which $\rho_n(\tau)$ = autocorrelation function of the noise.

By a change of variables $t = t$, $\theta = t - \tau$, the integral of Eq. 22 becomes:

$$E(C_n^2) = 1/T^2 \int \int A(t) A(t-\tau)\, \rho_n(\tau)\, dt\, d\tau = 1/T \int d\tau\, \rho_n(\tau) \left[1/T \int A(t) A(t-\tau) dt\right]$$

(Eq. 24)

The integral between the brackets in Eq. 24 is the time-autocorrelation function $R_S(\tau)$ of the expected signal A(t). Due to the flat spectrum of A(t), its autocorrelation function is approximately a sin x/x function. Since $A(t) = 0$ for $|t|>\frac{1}{2}T$, the correlation function $R_s(\tau) = 0$ for $|t|>T$. The limited time duration of the signal causes a tapering or $R_s(\tau)$ for larger values of τ. Therefore, if $T\Delta F >>1$, we may approximate the time-autocorrelation function by

$$R_S(\tau) = 1/T \int A(t)A(t-\tau)dt = \sigma_s^2\left[1-(|\tau|/T)\right]\left[(\sin\pi\Delta F\tau/\pi\Delta F\tau)\right] d\tau \quad \text{(Eq.25)}$$

Equation 24 now becomes

$$E\left[C_n^2\right]=1/T\int \rho_n(\tau)\, R_S(\tau)d\tau=1/T\int_{-T}^{T}\sigma_n^2(\sin\pi\Delta F\tau/\pi\Delta F\tau)\sigma_s^2\left[1-(|\tau|/T)\right] (\sin\pi\Delta F\tau/\pi\Delta F\tau) \quad d\tau \qquad \text{(Eq. 26)}$$

Since the contribution to the integral is small for large values of τ (assuming that $T\Delta F >>1$), the tapering factor $\left[1-(|t|/T)\right]$ will be neglected; so that

$$E(C_n^2) = 1/T \int_{-T}^{T} \sigma_n^2\, \sigma_s^2 \left[(\sin\, \pi\Delta F\,\tau/\,\tau\Delta F\,\tau)\right]^2 d\tau =$$

$$=(1/T\,\Delta F)\, \sigma_n^2\sigma_s^2\; (1/\pi) \int_{-\pi T\Delta F}^{\pi T\Delta F} \left[(\sin\,\pi\Delta F\,\tau/\pi\Delta F\,\tau)\right]^2 d(\pi\Delta F\,\tau)$$

(Eq. 27)

The integral limits may now be extended from minus to plus infinity, since the tails of the integral contribute little to the result if $T \Delta F \gg 1$.

The value of the integral from $-\infty$ to $+\infty$ is equal to π, so that

$$E(C_n^2) = (1/T\ \Delta F)\ \sigma_n^2 \sigma_s^2 = \boxed{1/N\ (\sigma_n^2 \sigma_s^2)} \qquad \text{(Eq. 28)}$$

Equation 28 gives the statistical average of the square of the correlator output, if the input is white noise with power equal to σ_n^2 and bandwidth ΔF. Because of the property of ergodicity, this statistical average is equal to the time average or power of the output noise.

c. <u>Signal-to-Noise Ratio</u>

From Eqs. 20 and 28 it is possible to determine the SNR of the output:

$$(SNR)_A = \left[a^2 \sigma_s^4 / (1/N) \sigma_n^2 \sigma_s^2\right] = N(a^2 \sigma_s^2 / \sigma_n^2) \qquad \text{(Eq. 29)}$$

It has been shown in the beginning of this chapter that the output SNR of the complete complex correlation is equal to the value found in Eq. 29 for the correlation of the real signal components only. Although the calculation has been made for $\varphi_0 = 0$, the result is also valid if $\varphi_0 \neq 0$.

Thus

$$(SNR)_{ref} = \boxed{N(a^2\sigma_s^2/\sigma_n^2)} = N.(SNR)_{in} \qquad \text{(Eq. 30)}$$

See also Eq. 11.

5. THE SAMPLED REFERENCE SYSTEM

Sampling the complex input signal is equivalent to sampling its quadrature components synchronously.

If the sampling interval is Δt, the sampled cosine component (indicated by a dot on the non-sampled function) may be represented by

$$\dot{\alpha}(t) = \Sigma\, \alpha(t_k)\ \Delta t\ \delta(t - t_k) \qquad \text{(Eq. 31)}$$

The scaling factor Δt is introduced to give the sampled signal the same impulse as the non-sampled signal. Correlation with $A(t)$ gives

$$C_{\dot{\alpha}}(\tau) = (1/T) \int \dot{\alpha}(t) A(t - \tau) dt = (1/T) \sum_k \alpha(t_k) \Delta t \int A(t-\tau)\ \delta(t-t_k)\ dt$$

$$= (1/T) \sum_k \alpha(t_k)\ A(t_k - \tau)\ \Delta t \qquad \text{(Eq. 32)}$$

5.1 Signal Correlation Peak

If $\varphi_o = 0$ and if the signals are in coincidence ($\tau = 0$), the correlation peak is

$$C_{\dot{\alpha}}(0) = (1/T) \sum_k \left[a\ A(t_k) + n(t_k) \right] A(t_k)\ \Delta t \qquad \text{(Eq. 33)}$$

The mean value is

$$E\left[C_{\alpha}^{\bullet}(0)\right] = C_{s}^{\bullet}(0) + E\left[C_{n}^{\bullet}(0)\right] = (a/T)\sum_{k} A^2(t_k)\Delta t + (1/T)\sum_{k} E\left[n(t_k)\right] A(t_k)\ \Delta t$$

(Eq. 34)

The statistical average of each noise sample $E\left[n(t_k)\right]$ equals zero, so that $E\left[C_{n}^{\bullet}(0)\right] = 0$ and

$$E\left[C_{\alpha}^{\bullet}(0)\right] = C_{s}^{\bullet}(0) = (a/T)\sum_{k} A^2(t_k)\ \Delta t \qquad \text{(Eq. 35)}$$

If the number of samples is large enough, the value of $\sum_{k} A^2(t_k)\Delta t$ is approximately equal to $\int A^2(t)\ dt = \sigma_s^2 T$ = energy of the expected signal. The actual value depends slightly on the position of the sampling comb.

The correlation peak (Eq. 35) of the sampling system will thus have the value $a\sigma_s^2$, as did the peak of the non-sampling system, (Eq. 20).

5.2 Noise

The power of the output noise is found by determining the variance

of the correlator output if the input is noise only.

$$E\left[C_n^{\bullet 2}(t_1)\right] = (1/T^2)\sum_k\sum_\ell A(t_k)\ A(t_\ell)\ E\left[n(t_k)\ \ n(t_\ell)\right]\Delta t^2 \qquad \text{(Eq. 36)}$$

with

$$E\left[n(t_k)n(t_\ell)\right] = \rho_n(t_k - t_\ell) = \rho_n(p\ .\ \Delta t)$$

$\rho_n(\tau) = \sigma_n^2\ (\sin \pi\cdot\Delta F\cdot\tau/\pi\cdot\Delta F\cdot\tau)$ = autocorrelation function of the input noise

By changing variables $t_k = t_k,\ t_\ell = t_k - p\,\Delta t$, Eq. 36 becomes

$$E\left[C_n^{\bullet 2}\right] = (1/T^2)\sum_p\sum_k A(t_k)\ A(t_k - p\,\Delta t\,)\ \rho_n(p\,\Delta t\,)\ \Delta t^2 =$$

$$=(1/T^2)\sum_p \Delta t\ \ \rho_n(p\,\Delta t)\left[1/T\sum_k A(t_k)\ A(t_k - p\,\Delta t)\ \Delta t\right] \qquad \text{(Eq. 37)}$$

The sum between the brackets in (Eq. 37) is the time-autocorrelation function $R_s^{\bullet}(p\,\Delta t)$ of the sampled expected signal. Although depending on the position of the sampling comb, its values are approximately equal to the corresponding values of the time-autocorrelation function $R_s(p\,\Delta t)$ of the non-sampled signal (Eq. 25).

Thus:

$$R_{\dot{s}}(p\ \Delta t) = \sigma_s^2 \left[1-(|p\ \Delta t|/T)\right] (\sin \pi\, \Delta F\, p\, \Delta t)/\pi\, \Delta F\, p\, \Delta t \qquad \text{(Eq. 38)}$$

where $|p\ \Delta t| < T$

If the tapering factor $1-(|p\ \Delta t|/T)$ is neglected, Eq. 37 becomes

$$E\left[C_{\dot{n}}^2\right] = (1/T)\sum_p \rho_n(p\ \Delta t)\, R_{\dot{s}}(p\ \Delta t)\, \Delta t = (\sigma_n^2\, \sigma_s^2/T)\sum_p \left(\sin \pi\, \Delta F\, p\, \Delta t/\pi\, \Delta F\, p\, \Delta t\right)^2 \Delta t$$

(Eq. 39)

For sampling frequencies that are multiples of ΔF, the sampling interval is

$$\Delta t = 1/(m\ \Delta F) \qquad (m \text{ integer}) \qquad \text{(Eq. 40)}$$

Equation 39 then becomes

$$E\left[C_{\dot{n}}^2\right] = \left(\sigma_n^2\, \sigma_s^2/m\, T\, \Delta F\right)\sum_p \left[\sin(p/m)\pi \big/ p/m)\pi\right]^2 = \left(\sigma_n^2 \sigma_s^2/N\right)\cdot\left(1/m\right) \sum_{-mN}^{mN} \left[\sin(p/m)\pi \big/ (p/m)\pi\right]^2 \qquad \text{(Eq. 41)}$$

The limits of p are found from the condition $|p|\Delta t<T \rightarrow |p|<mT\ \Delta F = mN$. The sum may be taken from $-\infty$ to $+\infty$, since the tails contribute little to its value if $N>>1$.

Since

$$\sum_{n=1}^{\infty} \sin^2 nx/n^2 = \tfrac{1}{2}x(\pi - x) \qquad 0 \leqslant x \leqslant \pi \qquad \text{(Ref. 4)} \qquad \text{(Eq. 42)}$$

it follows that

$$1/m \sum_{p}^{-\infty \ \infty} \left[\sin p(\pi/m) \Big/ p(\pi/m)\right]^2 = 1/m\left[\left(1+2m^2/\pi^2 \sum_{p=1}^{\infty} \left(\sin^2 p(\pi/m) \Big/ p^2\right)\right]\right.$$

$$= 1/m\left(1+\left[2m^2/\pi^2\right]\left[\tfrac{1}{2}\pi/m(\pi-\pi/m)\right]\right) = 1/m\left[1+(m-1)\right] = 1 \qquad \text{(Eq. 43)}$$

$$m \geqslant 1$$

The power of the noise at the output of the sampling system then follows from Eq. 41 and equals $\boxed{1/N\left(\sigma_n^2\,\sigma_s^2\right)}$, as in the case of the non-sampling system (Eq. 28).

5.3 Signal-to-Noise Ratio

Since neither the correlation peak nor the output noise is affected by the sampling, the output SNR of the sampled reference system is equal to the output SNR out of the reference system itself. (See also Appendix B of Ref. 5).

$$(\mathrm{SNR})_{\mathrm{ref.sampl.}} = \boxed{N\left(a^2\sigma_s^2/\sigma_n^2\right)} = N.(\mathrm{SNR})_{\mathrm{in}} \qquad \text{(Eq. 44)}$$

6. NORMALIZED MODULUS, NO PHASE QUANTIZING

The first type of quantizing of the received signal to be considered discards all amplitude information by normalizing the modulus of the complex signal, while the phase information is retained completely.

Technically this type of quantizing occurs when a high-frequency, narrow-band signal is clipped. After the clipping it is possible – by filtering and analogue quadrature demodulation – to extract the phase of the modulating low-frequency function with complete loss of amplitude information. The accuracy of the phase approximation is a function of the relative bandwidth $Q = F/\Delta F$ of the high-frequency signal. The results of this chapter are valid for a technical system in which $F/\Delta F >> 1$.

If the modulus of the received complex signal is normalized to $S = \sigma_s\sqrt{2}$, we may write (Fig. 4)

$$\xi_{norm}(t) = S\ e^{j\mathrm{arc}\,\xi(t)}$$

$$\text{with} \quad \xi(t) = aS\ e^{j\varphi_s(t)} + \nu(t) \qquad \text{(Eq. 45)}$$

$$\nu(t) = N(t)\ e^{j\varphi_n(t)} = \text{input noise}$$

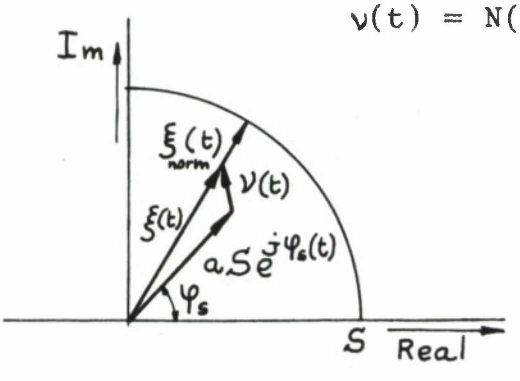

Fig. 4

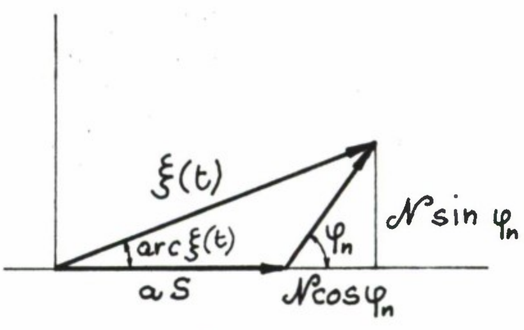

Fig. 5

6.1 Signal Correlation Peak

The mean value of the correlation peak is

$$E\left[\gamma(0)\right] = E\left[1/T \int \xi_{norm}(t) \quad S\, e^{-j\varphi_s(t)}\right] dt$$

$$= S^2/T \int E\left[e^{j\,\mathrm{arc}\,\xi(t)}\right] e^{-j\varphi_s(t)}\, dt \qquad \text{(Eq. 46)}$$

Since the noise vector $\nu(t)$ has a uniform phase distribution, the phase of $E\left[\xi_{norm}(t)\right]$ will be equal to the phase $\varphi_s(t)$ of the expected signal. By pairing positive and negative values of the phase φ_{noise} around $\varphi_s(t)$, it follows for a given value of $N(t)$ (see Fig. 5) that

$$E\left[e^{j\,\mathrm{arc}\,\xi(t)}\,\middle|\,N(t)\right] = e^{j\varphi_s(t)}\; E\left[\cos\left(\mathrm{arc}\,\xi(t)\right)\,\middle|\,N(t)\right]$$

$$= e^{j\varphi_s(t)} \cdot \frac{1}{\pi} \int_{\varphi_n=0}^{\pi} \frac{aS + N\cos\varphi_n}{\sqrt{a^2S^2 + 2aSN\cos\varphi_n + N^2}}\, d\varphi_n = e^{j\varphi_s(t)}\, I\left(\frac{aS}{N}\right)$$

(Eq. 47)

where

$$I\left(\frac{aS}{N}\right) = \frac{1}{\pi} \int_0^{\pi} \frac{aS/_N + \cos\varphi_n}{\sqrt{\left(aS/_N\right)^2 + 1 + 2\left(aS/_N\right)\cos\varphi}}\, d\varphi \qquad \text{(Eq. 48)}$$

Equation 48 can be solved using complete elliptic integrals, but the limiting cases of $\frac{aS}{N} \ll 1$ and $\frac{aS}{N} \gg 1$ are readily found directly:

$$I\left(\frac{a \cdot S}{N}\right) = \begin{cases} \frac{1}{2}\frac{aS}{N} & \text{if } \frac{aS}{N} \ll 1 \\ 1 - \frac{1}{2}\frac{1}{\left(aS/_N\right)^2} & \text{if } \frac{aS}{N} \gg 1 \end{cases} \qquad \text{(Eq. 49)}$$

To find the average $E\left[e^{jarc\ \xi(t)}\right]$, the integral in Eq. 48 has to be averaged over $N(t)$, ranging from 0 to ∞ with a Rayleigh distribution. Thus

$$E\left[e^{jarc\ \xi(t)}\right] = \int_0^\infty dN \cdot \frac{N}{\sigma_n^2} e^{-\frac{N^2}{2\sigma_n^2}} I\left(\frac{aS}{N}\right) \qquad \text{(Eq. 50)}$$

Substitution in Eq. 46 yields the mean value of the complex correlation peak of the modulus-normalizing system. For the limiting cases:

$$E\left[\gamma(0)\right] = \begin{cases} \sqrt{\pi}\ \sigma_s^2\ \frac{a\sigma_s}{\sigma_n} & \text{if } \frac{a\sigma_s}{\sigma_n} \ll 1 \\ 2\ \sigma_s^2\ \left(1 - \frac{1}{2}\frac{1}{\left(a\sigma_s/\sigma_n\right)^2}\right) & \text{if } \frac{a\sigma_s}{\sigma_n} \gg 1 \end{cases} \qquad \text{(Eq. 51)}$$

If the system is sampled, the same results hold.

6.2 Noise

In the absence of signal, the correlator output at a given instant t_1 equals

$$\gamma_n(t_1) = \frac{1}{T} \int \xi_{norm}(t)\ S\, e^{-j\varphi_s(t)} dt = \frac{S^2}{T} \int e^{j\varphi_n(t)}\ e^{-j\varphi_s(t)}\ dt$$

(Eq. 52)

This integral is a complex random variable with zero mean, since the mean of the normalized noise vector $e^{j\varphi_n(t)}$ equals zero.

The output power is equal to the variance of the complex variable (Eq. 52)

$$E\left[|\gamma_n|^2\right] = E\left[\gamma_n\ \gamma_n^*\right] = \frac{S^4}{T^2} \iint E\left[e^{j\left[\varphi_n(t)-\varphi_n(\theta)\right]}\right] e^{-j\left[\varphi_s(t)-\varphi_s(\theta)\right]}\ dt\ d\theta$$

$$= \frac{S^4}{T} \int d\tau\ E\left[e^{j\left[\varphi_n(t)-\varphi_n(t-\tau)\right]}\right] \left[\frac{1}{T} \int e^{-j\left[\varphi_s(t)-\varphi_s(t-\tau)\right]}\ dt\right] \quad \text{(Eq. 53)}$$

with $\tau = t - \theta$

The integral between brackets in the last expression of Eq. 53 is the conjugate complex time-autocorrelation function of the expected complex signal $e^{j\varphi_s(t)}$. This is a $\sin \pi\ \Delta F\ \tau/\pi\ \Delta F\ \tau$ function with zero phase, since the energy spectrum of the expected signal is a real, flat band of $\pm\frac{1}{2}\Delta F$ around zero frequency. See also Ref. 5 Appendix A. The tapering factor $\left[1-(|\tau|/T)\right]$ due to the limited

time-duration of the signal will be neglected, as in the foregoing chapters. Also, the integration limits $-T\ \Delta F$ to $+T\ \Delta F$ will be extended to $-\infty$ to $+\infty$, since the tails of the integration contribute little to the solution of Eq. 53 if $T\ \Delta F = N >> 1$. Thus

$$E\left[\left|\gamma_n\right|^2\right] = \frac{S^4}{N} \int_{-\infty}^{\infty} d(\Delta F\ \tau)\ \frac{\sin \pi\ \Delta F\ \tau}{\pi\ \Delta F\ \tau}\ \rho_{n\ norm}(\tau) \qquad \text{(Eq. 54)}$$

$\rho_{n\ norm}(\tau) = E\left[e^{j\left[\varphi_n(t) - \varphi_n(t - \tau)\right]}\right]$ is the auto-correlation function of complex Gaussian noise normalized to unit modulus. The correlation function may be written as

$$\rho_{n\ norm}(\tau) = E\left[e^{j(\varphi_1 - \varphi_2)}\right] = \int_{\varphi_1=0}^{2\pi} \int_{\varphi_2=0}^{2\pi} e^{j(\varphi_1-\varphi_2)}\ p(\varphi_1, \varphi_2)\ d\varphi_1\ d\varphi_2$$

$$= \int_{\varphi_2=0}^{2\pi} d\varphi_2 \int_{\psi=\varphi_2}^{2\pi-\varphi_2} e^{j\psi}\ p\left[\psi + \varphi_2, \varphi_2\right]\ d\psi\ , \qquad \text{(Eq. 55)}$$

where $\psi = \varphi_1 - \varphi_2$

and (Ref. 3 p. 164):

$$p\left[\varphi_1, \varphi_2\right] = \frac{\sigma_n^4 - \rho_n^2(\tau)}{4\pi^2\ \sigma_n^4}\ \left[\frac{(1-\beta^2)^{\frac{1}{2}} + \beta(\pi - \text{arc}\ \cos\ \beta)}{(1-\beta^2)^{3/2}}\right] = p(\psi) \qquad \text{(Eq. 56)}$$

in which

$$\beta = \frac{\rho_n(\tau)}{\sigma_n^2} \cos(\varphi_1 - \varphi_2) = \frac{\sin \pi \, \Delta F \, \tau}{\pi \, \Delta F \, \tau} \cos \psi$$

$$\rho_n(\tau) = \sigma_n^2 \frac{\sin \pi \, \Delta F \, \tau}{\pi \, \Delta F \, \tau}$$ = autocorrelation function of noise components

The function $p\left[\varphi_1, \varphi_2\right]$ is the joint probability density function of φ_1, φ_2 and is an even function $p(\psi)$ of $\varphi_1 - \varphi_2 = \psi$, as follows from (Eq. 56). If $\tau = 0$, Eq. 56 should reduce to

$$p_o(\psi) = \frac{1}{2\pi} \delta(\psi) \qquad \text{(Eq. 57)}$$

since the phase difference between the noise vectors is then zero with probability one.

The last integrand in Eq. 55 is periodic in ψ, so that the integration may be taken over the interval $(-\pi, +\pi)$:

$$\rho_{n\ norm}(\tau) = 2\pi \int_{-\pi}^{\pi} (\cos \psi + j \sin \psi) \, p(\psi) \, d\psi \qquad \text{(Eq. 58)}$$

The term with $\sin \psi$ will give zero, since $p(\psi)$ is an even function of ψ. Thus:

$$\rho_{n\ norm}(\tau) = 2\pi \int_{-\pi}^{\pi} \cos \psi \, p(\psi) \, d\psi \qquad \text{(Eq. 59)}$$

It follows from Eq. 57 that for $\tau = 0$

$$\rho_{n\ norm}(0) = \int_{-\pi}^{\pi} \delta(\psi)\, d\psi = 1\,, \qquad \text{(Eq. 60)}$$

as was to be expected.

The noise-output power becomes, from Eqs. 54, 56, & 59:

$$E\left[\,|Y_n|^2\,\right] = \frac{S^4}{N} \int_{-\infty}^{\infty} dx\ \mathrm{sinc}\ x\ (1 - \mathrm{sinc}^2\ x)\ \cdot$$

$$\cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi\ \cos\psi\ \frac{(1-\mathrm{sinc}^2\ x \cos^2\psi)^{\frac{1}{2}} + \mathrm{sinc}\ x \cos\psi \left[\pi - \arccos(\mathrm{sinc}\ x \cos\psi)\right]}{(1 - \mathrm{sinc}^2\ x \cos^2\psi)^{3/2}}$$

(Eq. 61)

$$\text{with} \quad \begin{cases} \mathrm{sinc}\ x = \dfrac{\sin \pi x}{\pi x} & \text{(Ref. 2)} \\[2ex] x = \Delta F\ \tau & \end{cases}$$

The solution of Eq. 61 has not been found, so that the SNR at the output of the modulus-normalizing system cannot be given.

If the system is sampled with a sampling interval

$$\Delta t = \frac{1}{m\ \Delta F} \qquad \text{(m integer)}$$

it can be shown, by following the same reasoning as in Chapter 5, that Eq. 54 takes the form

$$E\left[\,|Y_{\dot{n}}|^2\,\right] = \frac{S^4}{N} \cdot \frac{1}{m} \sum_{-mN}^{mN} \operatorname{sinc} \frac{p}{m} \; \rho_{n\ norm} \left(\frac{p}{m\ \Delta F}\right) \qquad \text{(Eq. 62)}$$

in which

$$\text{if} \quad p = 0 \qquad \rho_{n\ norm}(0) = 1 \qquad \text{(Eq. 63)}$$

$$\text{if} \quad p \neq 0 \qquad \rho_{n\ norm}\left(\frac{p}{m\ \Delta F}\right) = \left(1 - \operatorname{sinc}^2 \frac{p}{m}\right) \cdot$$

$$\cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} d\psi \cos\psi \; \frac{\left(1-\operatorname{sinc}^2\frac{p}{m}\cos^2\psi\right)^{\frac{1}{2}} + \operatorname{sinc}\frac{p}{m}\cos\psi\left[\pi - \arccos\left(\operatorname{sinc}\frac{p}{m}\cos\psi\right)\right]}{\left(1-\operatorname{sinc}^2\frac{p}{m}\cos^2\psi\right)^{3/2}}$$

From $m = 1$ (sampling with frequency ΔF), Eq. 62 becomes

$$E\left[\,|Y_{\dot{n}}|\,\right] = \frac{S^4}{N} \sum_{p=-N}^{N} \frac{\sin p\pi}{p\pi} \; \rho_{n\ norm}\left(\frac{p}{\Delta F}\right) \qquad \text{(Eq. 64)}$$

From Eq. 63 it follows that

$$\rho_{n\ norm}\left(\frac{p}{\Delta F}\right) = \begin{cases} 1 & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p \neq 0 \end{cases} \qquad \text{(Eq. 65)}$$

so that

$$E\left[\,|Y_{\dot{n}}|^2\,\right] = \frac{S^4}{N} = \boxed{\frac{4\sigma_s^4}{N}} \quad \text{(sampling freq.} = \Delta F\text{)} \qquad \text{(Eq. 66)}$$

6.3 Signal-to-Noise Ratio

The output SNR of the modulus-normalizing system, sampled with frequency ΔF, becomes (Eqs. 51 & 66)

$$N \cdot \frac{\pi}{4} \left(\frac{a\sigma_s}{\sigma_n}\right)^2 = 0.785 \cdot N \cdot (SNR)_{in} \qquad \text{if} \quad \frac{a\sigma_s}{\sigma_n} << 1$$

$(SNR)_{norm} =$ (Eq. 67)

$$N\left[1 - \frac{1}{2} \frac{1}{\left(a\sigma_s/\sigma_n\right)^2}\right]^2 = N\left[1 - \frac{1}{\left(a\sigma_s/\sigma_n\right)^2}\right] \qquad \text{if} \quad \frac{a\sigma_s}{\sigma_n} >> 1$$

It is seen that such a system is about 1 dB lower than the reference system for low-input SNR.

For high input SNR, the output SNR of the modulus-normalizing system saturates at a value N.

7. 90°-PHASE QUANTIZING

If four standard vectors making angles of 90° are used to quantize the complex input signal, the output properties will be independent of the orientation of the standard vector star. This is a consequence of the assumed uniform phase distribution of the signals and the complete phase description of the expected signal.

We will determine the output SNR of a system with the vector star orientated at 45° with respect to the axes of the complex plane. (Fig. 6).

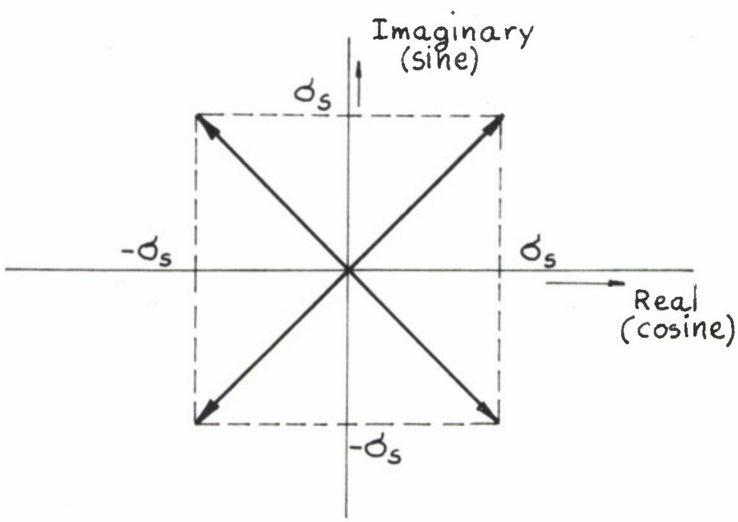

Fig. 6

The real and imaginary parts of the complex input signal can then only take the values $+\sigma_s$ and $-\sigma_s$ i.e. they are <u>infinitely clipped</u>.

The following notation will be used for the clipped components:

$$\hat{\alpha}(t) = \begin{cases} +\sigma_s & \text{if} \quad \alpha(t) > 0 \\ -\sigma_s & \text{if} \quad \alpha(t) < 0 \end{cases} \qquad \text{(Eq. 68)}$$

7.1 Signal Correlation Peak

If $\varphi_o = 0$, the cosine component of the input is

$$\alpha(t) = a\, A(t) + n(t)$$

The mean value of the correlation peak is

$$E\left[C_{\hat{s}}(0)\right] = \frac{1}{T} \int E\left[\hat{\alpha}(t)\right] A(t)\, dt \qquad \text{(Eq. 69)}$$

To determine $E\left[\hat{\alpha}(t)\right]$, we consider two cases, $A(t) > 0$ and $A(t) < 0$, it follows then that

$$A(t) > 0 \rightarrow E\left[\hat{\alpha}(t)\right] = (+\sigma_s)\text{Prob}\left[n(t) > -a\, A(t)\right] + (-\sigma_s)\text{Prob}\left[n(t) < -a\, A(t)\right]$$

$$A(t) < 0 \rightarrow E\left[\hat{\alpha}(t)\right] = (+\sigma_s)\text{Prob}\left[n(t) > |a\, A(t)|\right] + (-\sigma_s)\text{Prob}\left[n(t) < |a\, A(t)|\right]$$

(Eq. 70)

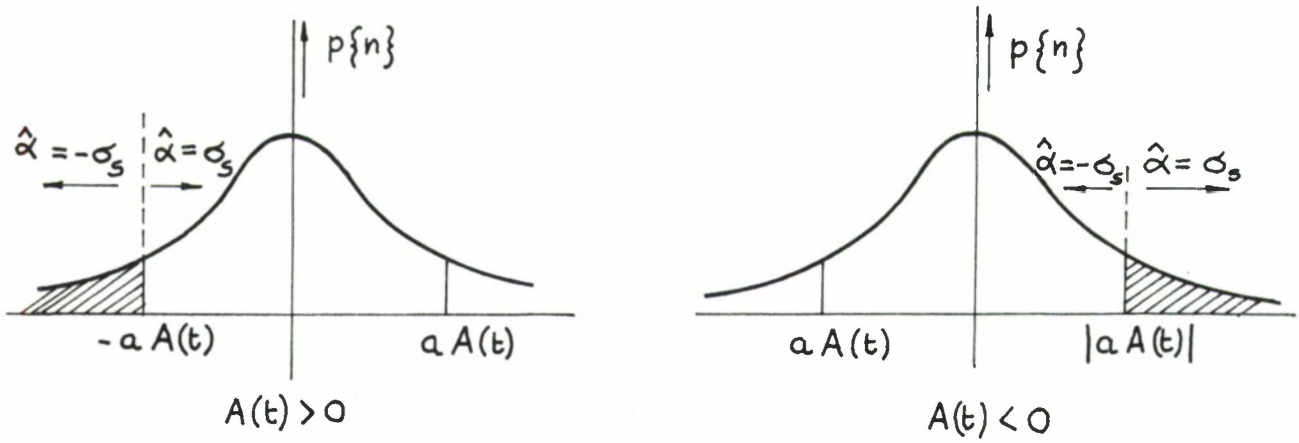

Fig.7

It follows from Eq. 70 and the Gaussian amplitude distribution of the noise (Fig. 7) that the probability that the mean value of the clipped signal at a given instant has the same sign as that of the expected signal A(t) is

$$I\left[\frac{a\,|A(t)|}{\sigma_n}\right] = \frac{1}{\sigma_n\sqrt{2\pi}} \int_{-a|A(t)|}^{a|A(t)|} e^{-\frac{n^2}{2\sigma_n^2}}\, dn = \sqrt{2/\pi} \int_{0}^{\frac{a|A(t)|}{\sigma_n}} e^{-\frac{1}{2}\xi^2}\, d\xi$$

(Eq. 71)

The mean value of the clipped signal thus becomes:

$$E\left[\hat{\alpha}(t)\right] = \frac{A(t)}{|A(t)|} \cdot \sigma_s\, I\left[\frac{a\,|A(t)|}{\sigma_n}\right] \qquad \text{(Eq. 72)}$$

Two limiting cases will be considered: low and high input SNR.

a. If $\frac{a|A(t)|}{\sigma_n} \ll 1$, it follows from a power series expansion that

$$I\left[\frac{a|A(t)|}{\sigma_n}\right] = \sqrt{2/\pi}\;\frac{a|A(t)|}{\sigma_n} \qquad \text{(Eq. 73)}$$

Thus, from Eq. 72,

$$E\left[\hat{\alpha}(t)\right] = \frac{A(t)}{|A(t)|}\,\sigma_s\,\sqrt{2/\pi}\;\frac{a|A(t)|}{\sigma_n} = \frac{a\sigma_s}{\sigma_n}\sqrt{2/\pi}\;A(t) \qquad \text{(Eq. 74)}$$

So that the correlation peak (Eq. 69) becomes

$$E\left[C_{\hat{s}}(0)\right] = \frac{a\sigma_s}{\sigma_n}\sqrt{2/\pi}\;\frac{1}{T}\int A^2(t)\,dt = \boxed{\sqrt{2/\pi}\;\frac{a\sigma_s}{\sigma_n}\;\sigma_s^2} \qquad \text{(Eq. 75)}$$

b. If $\frac{a|A(t)|}{\sigma_n} \gg 1$, then $I\left[\frac{a|A(t)|}{\sigma_n}\right] \approx 1$.

Consequently $E\left[\hat{\alpha}(t)\right] = \frac{A(t)}{|A(t)|}\,\sigma_s$, i.e. the signal-plus-noise has, with probability one, the same sign as the signal only, as is obvious. The mean value of the correlation peak becomes in this case, from Eq. 69:

$$E\left[C_{\hat{s}}(0)\right] = \frac{1}{T}\int \sigma_s\,\frac{A(t)}{|A(t)|}\,A(t)\,dt = \frac{\sigma_s}{T}\int |A(t)|\,dt \qquad \text{(Eq. 76)}$$

The last expression involves calculation of the average absolute value of the cosine component A(t) of the expected signal. Because of the uniform phase distribution and the constant amplitude, the average absolute value is equal to that of a sinusoid with the same amplitude. Thus

$$\frac{1}{T} \int |A(t)| dt = \frac{2}{\pi} \sigma_s \sqrt{2} \qquad \text{(Eq. 77)}$$

where $\sigma_s \sqrt{2}$ = amplitude of the signal component.
So that

$$E\left[C_{\hat{s}}(0)\right] = \boxed{\frac{2\sqrt{2}}{\pi} \quad \sigma_s^2} \qquad \text{(Eq. 78)}$$

If the input signal is not only quantized, but also sampled, the same results are valid. See also Chapter 5.

7.2 Noise

In the absence of signal, the correlator output at a given instant is

$$C_{\hat{n}}(t_1) = \frac{1}{T} \int \hat{n}(t)\ A(t)\ dt \qquad \text{(Eq. 79)}$$

This equation has the same form as Eq. 21. The variance of the random variable (Eq. 79) can be determined by following the same reasoning as in Chapter 4.2.

Taking into account that the autocorrelation function of the clipped Gaussian noise is

$$\rho_{\hat{n}}(\tau) = \frac{2}{\pi} \sigma_s^2 \operatorname{arc\,sin}\left[\frac{\rho_n(\tau)}{\sigma_n^2}\right] = \frac{2}{\pi} \sigma_s^2 \operatorname{arc\,sin}\left[\frac{\sin \pi \Delta F \tau}{\pi \Delta F \tau}\right], \qquad \text{(Eq. 80)}$$

the expectation of the squared correlator output for noise only becomes (Eqs. 22 to 27):

$$E\left[C_{\hat{n}}^2\right] = \frac{1}{T \Delta F} \sigma_s^4 \int_{-T\Delta F}^{T\Delta F} \frac{\sin \pi \Delta F \tau}{\pi \Delta F \tau} \frac{2}{\pi} \operatorname{arc\,sin}\left[\frac{\sin \pi \Delta F \tau}{\pi \Delta F \tau}\right] d(\Delta F \tau)$$

(Eq. 81)

The integral in the expression above has been calculated to be 0.7898 for the limits $-\infty$ to $+\infty$. The noise power at the correlator output becomes

$$E\left[C_{\hat{n}}^2\right] = \boxed{0.7898 \frac{1}{N} \sigma_s^4} \qquad \text{(Eq. 82)}$$

If the system is sampled with a sampling interval of

$$\Delta t = \frac{1}{m \Delta F} \text{ (m integer)},$$

Eq. 81 takes the form

$$E\left[C_{\hat{n}}^2\right] = \frac{1}{N} \sigma_s^4 \frac{1}{m} \sum_{p=-mN}^{mN} \frac{\sin \frac{p}{m} \pi}{\frac{p}{m} \pi} \cdot \frac{2}{\pi} \operatorname{arc\,sin}\left[\frac{\sin \frac{p}{m} \pi}{\frac{p}{m} \pi}\right] = k \frac{\sigma_s^4}{N}$$

with $N = T.\Delta F$ (Eq. 83)

The results in Table 1 were found for the factor k in Eq. 83 when $N>>1$:

m	$\frac{1}{m}\sum_{-\infty}^{\infty} = k$	$\frac{1}{k}\ \frac{2}{\pi}$	$\frac{1}{k}\ \frac{\pi}{4}$	$\frac{1}{k}\ \frac{8}{\pi^2}$	$\frac{1}{k}$
1	1.000	0.6366	0.7855	0.8105	1.0000
2	0.8402	0.7577	0.9348	0.9648	1.1902
3	0.8117	0.7844	0.9676	0.9986	1.2320
4	0.8021	0.7937	0.9794	1.0106	1.2467
8	0.7929	0.8030	0.9906	1.0223	1.2612
16	0.7906	0.8052	0.9935	1.0252	1.2648
32	0.7901	0.8058	0.9940	1.0260	1.2657
64	0.7899	0.8060	0.9942	1.0262	1.2660
128	0.7898	0.8062	0.9945	1.0264	1.2663

<u>TABLE 1</u>

Noise figure k and output SNR for low and high input SNR as a function of the sampling rate

It is seen from Table 1 that the noise figure

$$k = \frac{1}{m}\sum_{p=-\infty}^{\infty} \frac{\sin \frac{p}{m}\pi}{\frac{p}{m}\pi} \cdot \frac{2}{\pi} \arcsin \left[\frac{\sin \frac{p}{m}\pi}{\frac{p}{m}\pi}\right] \quad \text{(Eq. 84)}$$

converges rapidly to an asymptotic value of about 0.7898, as given in Eq. 82 for the solution of the integral in Eq. 81.

7.3 Signal-to-Noise Ratio

The output SNR of the 90°-phase quantizing correlator is found from Eqs. 75, 78, & 83 and Table 1.

For low and high input SNR:

$$(SNR)_{out} = \begin{cases} \frac{1}{k} \cdot \frac{2}{\pi} \cdot N \cdot \left(\frac{a\sigma_s}{\sigma_n}\right)^2 & \text{if } (SNR)_{in} \ll 1 \\ \\ \frac{1}{k} \frac{8}{\pi^2} \cdot N & \text{if } (SNR)_{in} \gg 1 \end{cases} \qquad \text{(Eq. 85)}$$

It follows from Table 1 that, for normal sampling (ΔF) resp. infinite sampling the losses as compared to the reference system vary from 2 dB to 1 dB for low input SNR. For high input SNR the performance under normal sampling is 1 dB less than that of a correlator with no phase quantizing (Chapter 6.3).

The behaviour of the non-sampling system cannot be compared to the non-sampling, non-phase quantizing system of Chapter 6, since the solution of Eq. 61 has not been found.

8. $\frac{\pi}{m}$ -PHASE QUANTIZING

If the complex input signal is regularly phase-quantized to angles smaller than 90°, it is for some cases possible to determine the effects of this finer quantizing on the signal correlation peak and on the noise.

8.1 Signal Correlation Peak

Independent of the sampling frequency, the mean signal correlation peak can be determined directly if the input SNR is high. In that case the received complex signal will fall, with probability one, in the sector belonging to the expected complex signal (Fig. 8).

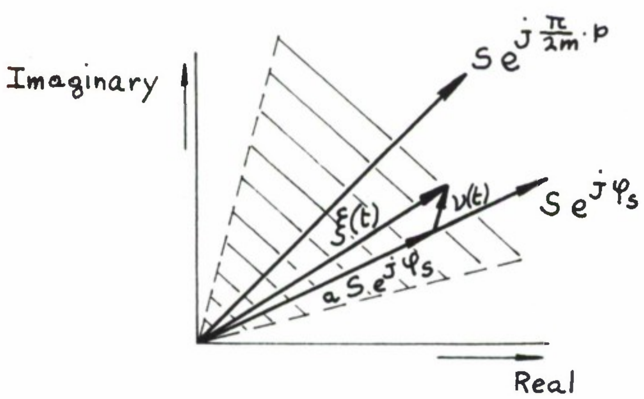

Fig. 8

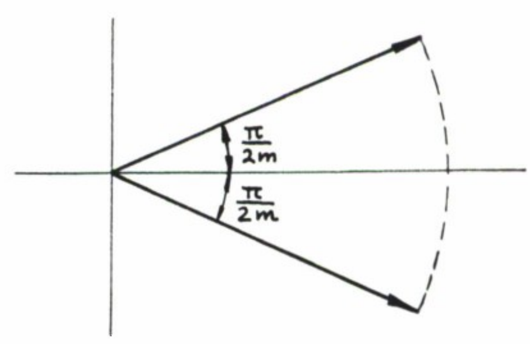

Fig. 9

The phase difference between the quantized received vector and the non-quantized expected signal vector ranges therefore from $-\frac{1}{2}\frac{\pi}{m}$ to $+\frac{1}{2}\frac{\pi}{m}$. (See Fig. 9). On the average, the length in the direction of the correlation peak (phase angle zero, since it is assumed that $\varphi_o = 0$) becomes smaller with a factor:

$$\frac{1}{\frac{\pi}{m}} \int_{-\frac{\pi}{2m}}^{\frac{\pi}{2m}} \cos\varphi \, d\varphi = \frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}} \qquad \text{(Eq. 86)}$$

The modulus-normalizing system of Chapter 6 should be taken as a reference. It was shown there (Eq. 51) that for high input SNR the signal correlation peak saturates at a value $2\sigma_s^2$.

Thus the mean value of the correlation peak is

$$E\left[\gamma_{\hat{s}}(0)\right] = \boxed{2\sigma_s^2 \frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}} \qquad \text{(Eq. 87)}$$

As an example we consider the case m = 2, i.e. 90°-phase quantizing.

Then

$$E\left[\gamma_{\hat{s}}(0)\right] = 2\sigma_s^2 \frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}} = 2\sigma_s^2 \frac{2\sqrt{2}}{\pi} \qquad \text{(Eq. 88)}$$

Compare this result with Eq. 78, which is half its value because it gives the contribution of only one component to the correlation peak, (α A).

8.2 Noise

If the sampling frequency is ΔF, the complex noise samples are uncorrelated. Independent of the phase-quantizing, the output of the correlator will be the sum of N vectors with equal lengths and random orientation. For the limiting case of infinitely fine phase quantizing, the result has already been found in Eq. 66.:

$$E\left[|Y_{\hat{n}}^{\bullet}|^2 \right] = \frac{4\sigma_s^4}{N} \qquad \text{(Eq. 89)}$$

8.3 Signal-to-Noise Ratio

For a phase-quantizing correlator sampling with sampling frequency ΔF, the output signal SNR for high input SNR is

$$(\mathrm{SNR})_{out} = \boxed{N\left(\frac{\sin \frac{\pi}{2m}}{\frac{\pi}{2m}}\right)^2} \qquad \text{(Eq. 90)}$$

9. COMBINATION OF RESULTS

The results found in the foregoing chapters will now be combined and shown graphically as functions of the degree of phase-quantizing, the sampling interval, and the input signal-to-noise ratio.

9.1 Signal Correlation Peak

The power of the signal-output peak is represented in two three-dimensional drawings: Fig. 10 for low input SNR, Fig. 11 for high input SNR. A scaling factor $N/4\sigma_s^4$ is used to normalize the results of Chapters 6, 7, and 8.

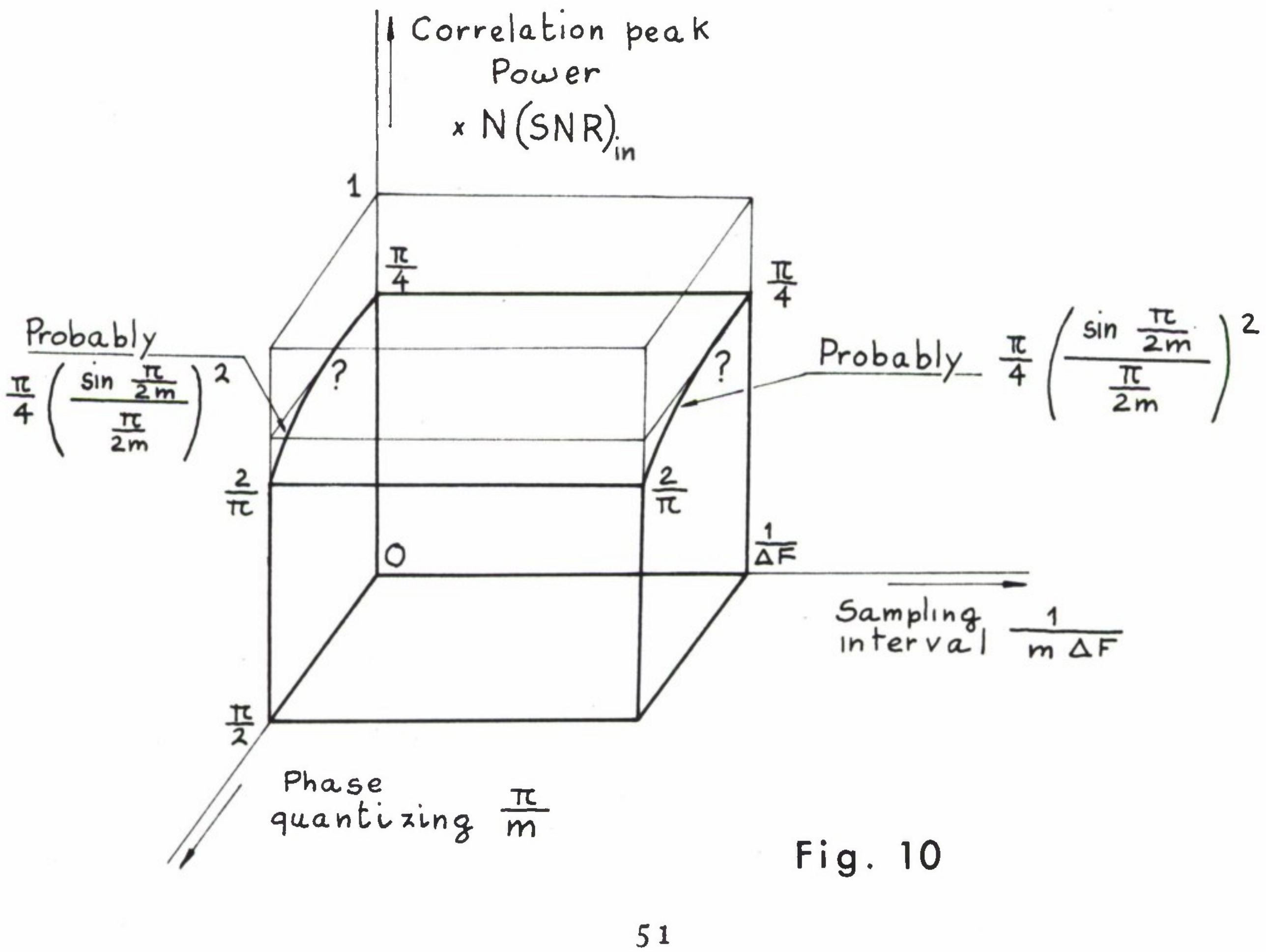

Fig. 10

For low input SNR, the signal peak power is independent of the sampling interval, varying from $\pi/4 \ N.(SNR)_{in}$ for no-phase quantizing, to $2/\pi \ N.(SNR)_{in}$ for 90°-phase quantizing. The intermediate values probably follow the function

$$\pi/4 \left[\sin(\pi/2m) \big/ (\pi/2m)\right]^2 \ N.(SNR)_{in}$$

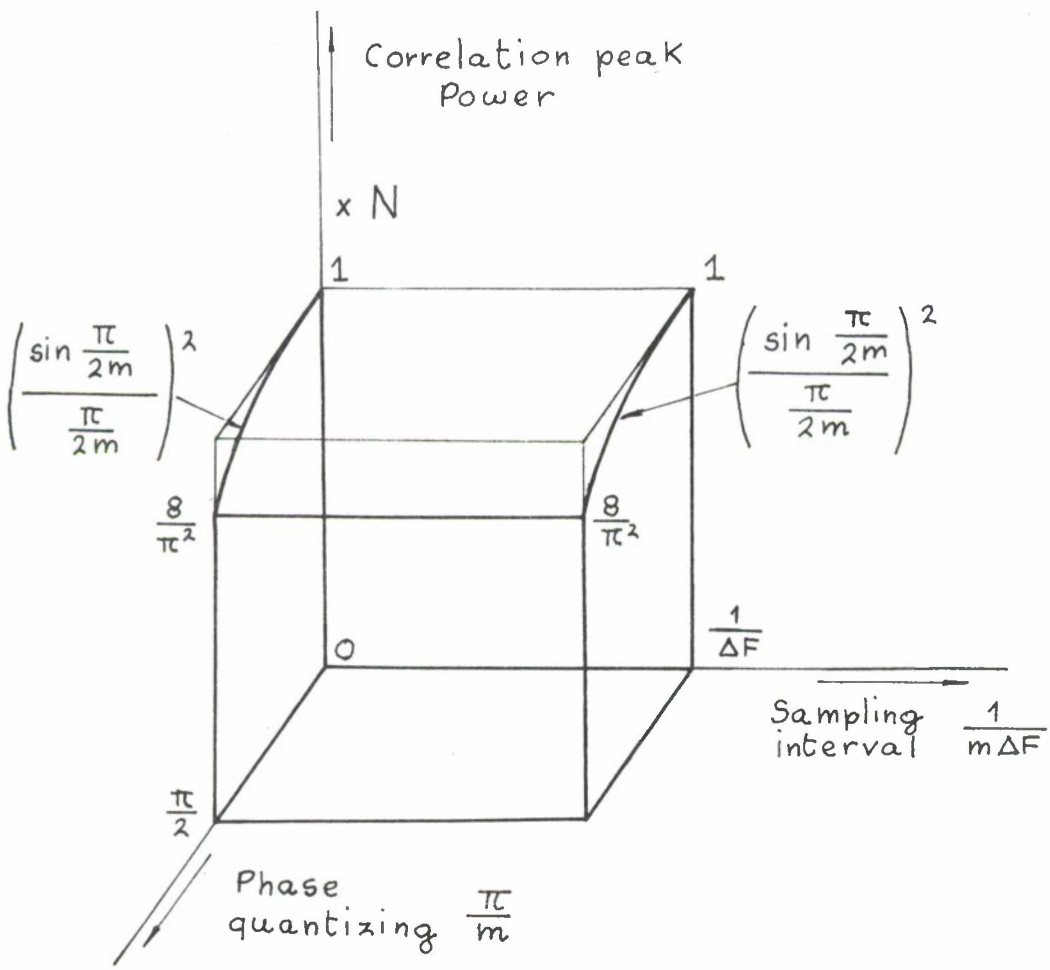

Fig. 11

For high input SNR, the signal peak power saturates to a value N in the case of no-phase quantizing (only modulus normalizing). With increasing phase quantizing, the signal level goes down according to a $\left[\sin(\pi/2m) \Big/ (\pi/2m)\right]$ - law, where π/m is the step of the phase quantizing. The output peak power does not depend on the sampling frequency. The scaling factor is $4\sigma_s^4$.

9.2 Noise

The output noise power for a modulus normalizing, phase quantizing, and sampling complex correlator is given in Fig. 12.

The values found in Chapters 6, 7, and 8 are multiplied by the scaling factor $N/4\sigma_s^4$ to make the results comparable to the signal values of function $\pi/4\left[\sin(\pi/2m \Big/ (\pi/2m)\right]^2 N(SNR)_{in}$

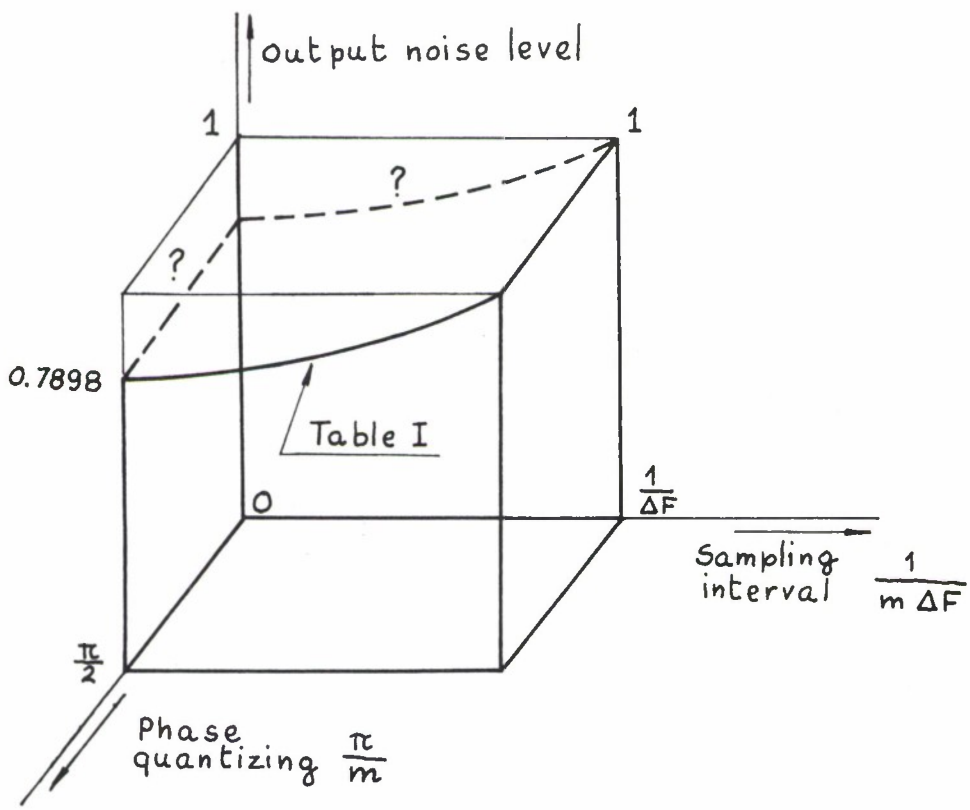

Fig. 12

In Chapter 8 it has been shown that when sampling with frequency ΔF, the noise output power is independent of the phase quantizing. Although not proved, it is assumed that for other sampling frequencies the output-noise power is also independent of the phase quantizing. This means that for no-phase quantizing (only modulus normalizing) the noise-output power as a function of the sampling rate is that given in Table 1 (dotted line in Fig. 12).

9.3 Signal-to-Noise Ratio

The output signal-to-noise ratios for low and high input SNR (Figs. 13 and 14) follow directly from the above paragraphs.

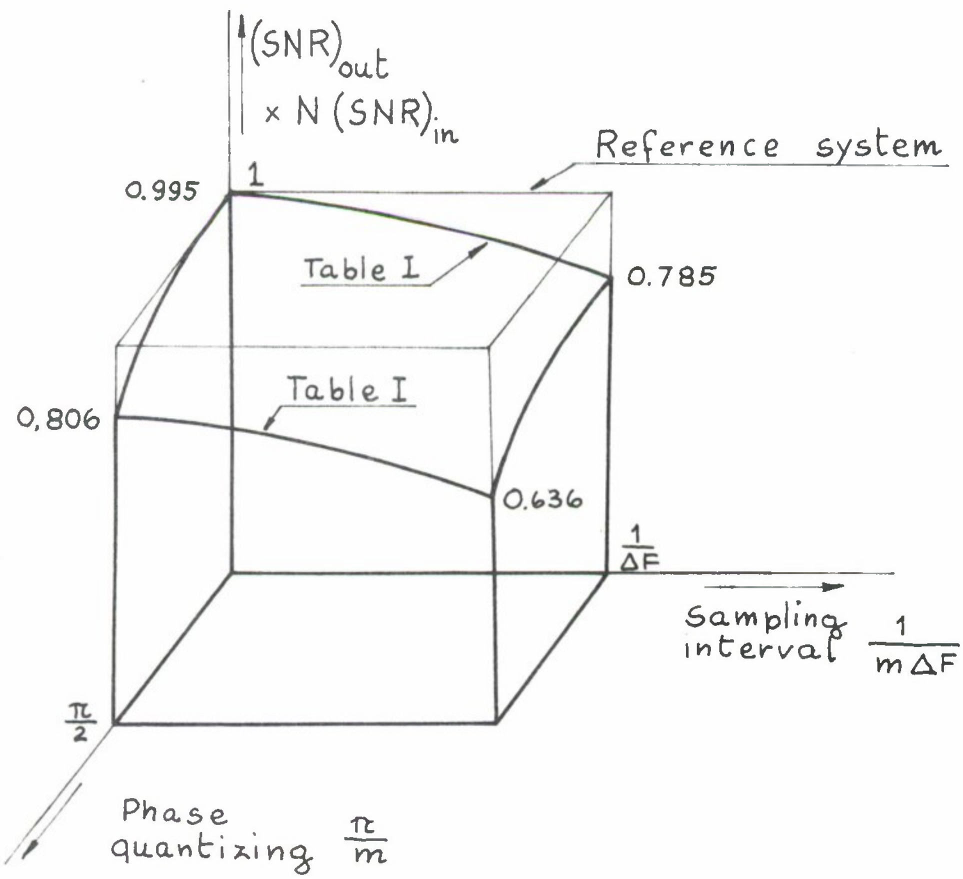

Fig. 13

If we accept the assumption that the output-noise power is independent of the phase quantizing, then the output SNR for modulus normalizing only is about equal to the output SNR of the reference system for low input SNR. Both quantizing and sampling of the complex input signal reduce the output SNR by about 1 dB. If the input signal is not only quantized by 90°, but also sampled with sampling frequency ΔF, the total loss is about 2 dB.

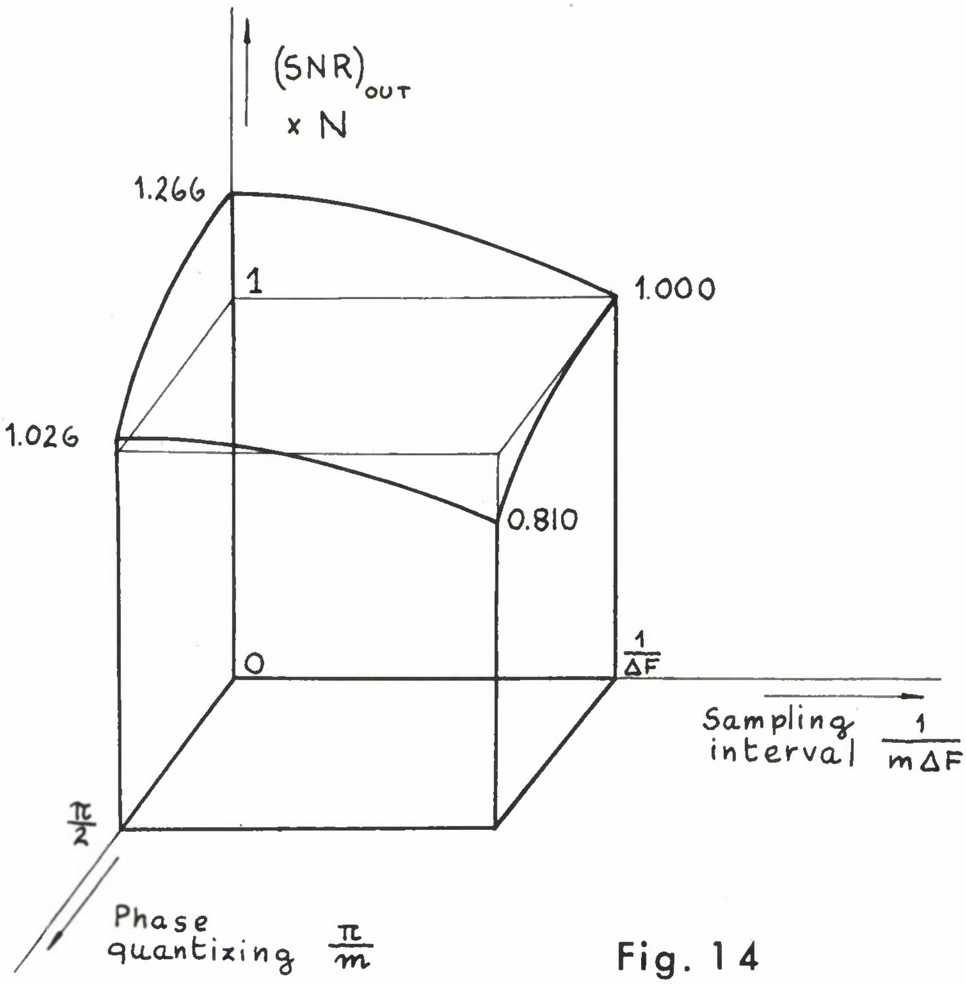

Fig. 14

For high input SNR, the output SNR saturates at a maximum value of 1.266 N (Table 1).

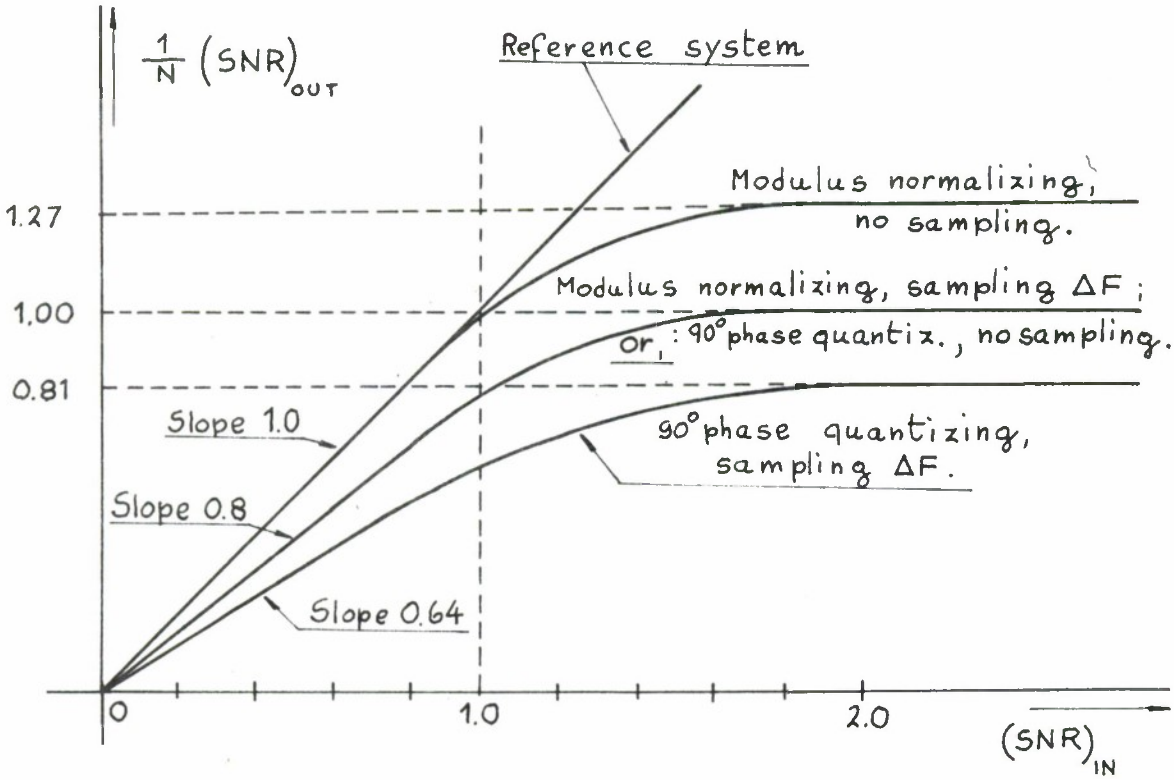

Fig. 15

In Fig. 15 the combined influences of modulus normalizing, phase quantizing, and sampling are given as functions of the input SNR. The curves are interpolated between low and high input SNR.

It follows from Fig. 15 that normalizing only the modulus causes saturation of the input SNR for high input SNR.

Sampling and no-phase quantizing, or 90°-phase quantizing and no sampling, makes the system about 1 dB less than the modulus normalizing system.

Both 90°-phase quantizing and sampling cause a total loss of 2 dB.

It has been shown in Ref. 5 that 90°-phase quantizing of the expected signal causes an additional loss of 1 dB. This would raise the losses to a total of 3 dB as compared with those of the modulus normalizing system without phase-quantizing of the expected signal, and without sampling.

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